Response of a Beam on an Inertial Foundation to a Traveling Load

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The response of a Bernoulli-Euler beam supported by a Winkler-type elastic foundation with inertia and subjected to a moving load is investigated. Steady-state solutions are determined for an undamped and linearly damped beam-foundation system. The effects on the response of load velocity, foundation mass, and damping are studied. For the undamped system, it is well known that the response grows without bound as a certain critical velocity is approached. It is shown that the effect of foundation mass is to reduce the critical velocity and to increase the peak deflection. The increase in peak deflection becomes more pronounced as the critical velocity is approached. As in the case of massless foundation, the deflection wave is observed to be symmetric with respect to the load. When damping is introduced, the deflection wave loses its symmetry, and the peak deflection is reduced. Results for both cases are given in graphical form.

Nomenclature

A = beam cross-sectional area
E = Young's modulus of beam material
E_f = Young's modulus of foundation
I = moment of inertia of beam
P = dimensionless concentrated traveling load
Re = real part of a complex quantity
U = dimensionless foundation displacement
V = load dimensionless velocity
W = dimensionless beam deflection
W_p = peak (dimensionless) beam deflection
W_s = peak static (dimensionless) beam deflection
c = viscous damping coefficient for beam
c_f = velocity of plane wave propagation in foundation, see Eq. (2)
g_n = constants, see Eqs. (21)
h = beam width
k_f = spring modulus of foundation (force/unit area)
m = beam density (mass per unit length)
q = foundation pressure (F/L)
r = radius of gyration of beam $(\sqrt{I/A})$
t = time
u = foundation displacement
v = load velocity
w = beam deflection
x = coordinate along beam axis
y = variable in characteristic equation, see Eq. (19)
z = coordinate normal to beam axis, see Eq. (2)
Ω = "bouncing frequency" of beam on massless foundation
α = ratio of foundation stiffness to beam stiffness $(mr^4\Omega^2/EI)$
β^2 = ratio of the mass-per-unit length of the foundation to the mass-per-unit-length of the beam $(\rho\ell/m)$
γ = dimensionless distance along beam axis $(\xi - V\tau)$
η = dimensionless distance along foundation, positive down (z/r)
θ = dimensionless damping coefficient $(c/2m\Omega)$
λ = separation constant for foundation equation, also variable in characteristic equation, see Eqs. (11), (17), and (18)
ξ = dimensionless distance along beam axis (x/r)
ρ = mass of foundation material per unit length of foundation per unit length of beam
τ = dimensionless time (Ωt)

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I. Introduction

THE dynamic response of a beam on an elastic foundation has been the object of investigation by several authors. One of the earliest studies was performed by Timoshenko.[1] His work was concerned with the response of a rail subjected to a constant load moving at constant velocity. He noted the possibility of a resonance condition for load velocities considerably in excess of velocities experienced at that time by vehicles traveling on rails. Somewhat later, Ludwig[2] and Dorr[3] investigated the moving load problem more thoroughly. None of the mathematical models used in these studies included the effect of damping or foundation mass on the response.

The advent of high-speed rocket-propelled sleds that move on rails made the resonance condition noted by Timoshenko somewhat less academic and resulted in a study by Kenney,[4] who included the effect of linear damping on the response. Shortly thereafter, Matthews,[5,6] using a slightly different approach, essentially duplicated Kenney's results.

In all these studies, the model considered was a Bernoulli-Euler beam of uniform density and stiffness supported by a massless Winkler-type foundation. It is known,[7,8] however, that the Bernoulli-Euler beam model becomes inaccurate at the high frequencies associated with the higher load velocities. With this fact in mind, Crandall[9] studied the moving load problem, utilizing the Timoshenko beam model that is applicable at the higher frequencies. Crandall noted that, in addition to the resonant condition shown by the Bernoulli-Euler beam model, the Timoshenko model exhibits two additional resonances. In a similar investigation, Achenbach and Sun[10] discussed the character of the resonances peculiar to the Timoshenko model.

Achenbach and Sun[11] analyzed the dynamic response of a beam on a viscoelastic subgrade. Their foundation model was obtained by replacing the springs of the Winkler foundation by viscoelastic elements. Rades[12] considered the steady-state response of a finite Bernoulli-Euler beam on a Pasternak-type foundation. In this study, some observations were made on the influence of damping and inertia of the foundation.

Recent emphasis on mass transit led to an investigation by Kaplan et al.[13] wherein the effect of foundation inertia was approximated by including a point mass under an oscillatory stationary load. Saito and Murakami[14] studied wave propagation in an infinite Timoshenko beam on a Winkler-type foundation with consideration of foundation mass. They concluded that the assumption of a massless foundation is good only for low frequencies and small foundation-to-beam mass ratios.

More recently, Rades[15] generalized the three-parameter massless foundation model proposed by Kerr[16] to include the effect of foundation inertia on the steady-state motion of a rigid beam.

The purpose of the present investigation is to study the response of an infinite Bernoulli-Euler beam supported by an inertial Winkler-type foundation and subjected to a moving concentrated load. The load is assumed to move with constant velocity. The effects on the system response of load velocity, damping, and foundation mass are investigated and results are given in graphic form.

II. Equations of Motion

The differential equation of motion for an Euler-Bernoulli beam is

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+m\frac{\partial^2 w(x,t)}{\partial t^2}+c\frac{\partial w(x,t)}{\partial t}=p(x,t)-q(x,t) \tag{1}$$

where p is the applied load, q is the foundation pressure, w is the beam displacement, and all other symbols have their usual meaning. Boundary conditions will be discussed later. According to the model proposed by Saito and Murakami,[14] the motion of the foundation is governed by the familiar one-dimensional wave equation

$$\frac{\partial^2 u(z,t)}{\partial t^2}=c_f^2\frac{\partial^2 u(z,t)}{\partial z^2} \tag{2}$$

where u is the foundation displacement. The foundation boundary conditions are

$$u(o,t)=w(x,t) \qquad u(\ell,t)=0 \tag{3}$$

Figure 1 shows the relationship between the beam and foundation coordinate systems.

It is convenient to rewrite the above equations in the following nondimensional form

$$\frac{EI}{mr^4\Omega^2}\frac{\partial^4 W}{\partial\xi^4}+\frac{\partial^2 W}{\partial\tau^2}+\frac{c}{m\Omega}\frac{\partial W}{\partial\tau}=\frac{p}{mr\Omega^2}-\frac{q}{mr\Omega^2} \tag{4}$$

$$\frac{\partial^2 U}{\partial\tau^2}=\frac{c_f^2}{r^2\Omega^2}\frac{\partial^2 U}{\partial\eta^2} \tag{5}$$

$$U(L,\tau)=0 \qquad U(o,t)=W(\xi,\tau) \tag{6}$$

where

$$\xi=x/r \qquad \eta=z/r \qquad L=\ell/r \qquad W=w/r$$

$$U=u/r \qquad \tau=\Omega t \qquad r=\sqrt{I/A} \qquad \Omega=\sqrt{k_f/m}$$

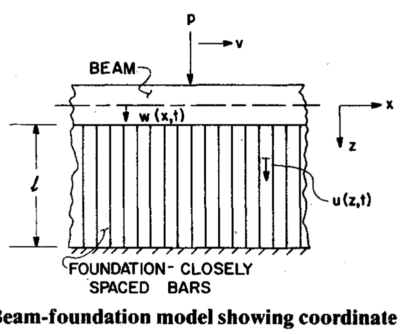

Fig. 1 Beam-foundation model showing coordinate system.

and

$$q=-k_f Lr\frac{\partial U}{\partial\eta}\bigg|_{\eta=0}$$

III. Solution of the Equations of Motion

Steady-state solutions will be sought. The technique to be used involves finding traveling solutions that satisfy the beam equations without the forcing function and the foundation equation. Linear combinations of these solutions are then used to satisfy the boundary conditions. Solutions of this type have been named steady state, since an observer moving with the load always sees the same deflection wave.

Assume that Eqs. (4) and (5) are satisfied by

$$W=F(\xi-V\tau) \tag{7}$$

$$U=D(\eta)F(\xi-V\tau) \tag{8}$$

where V is the load dimensionless velocity $v/r\Omega$.

Introducing Eqs. (7) and (8) into Eqs. (4) and (5) with $p=0$ and letting $\gamma=\xi-V\tau$ yields

$$\frac{1}{\alpha}\frac{\mathrm{d}^4F}{\mathrm{d}\gamma^4}+V^2\frac{\mathrm{d}^2F}{\mathrm{d}\gamma^2}-2\theta V\frac{\mathrm{d}F}{\mathrm{d}\gamma}=-\frac{1}{mr\Omega^2}q \tag{9}$$

$$V^2D(\eta)\frac{\mathrm{d}^2F}{\mathrm{d}\gamma^2}=\frac{c_f^2}{r^2\Omega^2}\frac{\mathrm{d}^2D(\eta)}{\mathrm{d}\eta^2}F \tag{10}$$

where $\alpha=mr^4\Omega^2/EI=r^4k_f/EI$, $\theta=c/2m\Omega$. Separation of variables in Eq. (10) leads to

$$\frac{\mathrm{d}^2F}{\mathrm{d}\gamma^2}+\lambda^2F=0 \tag{11}$$

$$\frac{\mathrm{d}^2D}{\mathrm{d}\eta^2}+\left(\frac{\lambda Vr\Omega}{c_f}\right)^2D=0 \tag{12}$$

It should be noted that the choice of the separation constant λ is arbitrary at this point. Let $\beta^2=\rho\ell/m$, the ratio of the foundation mass (per unit length of the beam) to the mass per unit length of the beam. Then $(r\Omega/c_f)^2=(\beta/L)^2$.

It is easy to show that the solution of Eq. (12), subject to $D(0)=1$, $D(L)=0$, is

$$U=\frac{\sin[\beta V\lambda(1-\eta/L)]}{\sin(\beta V\lambda)}F(\gamma) \tag{13}$$

Following Saito and Murakami,[14] the foundation pressure q can be expressed as

$$q=-k_f\ell\frac{\partial U}{\partial\eta}\bigg|_{\eta=0} \tag{14}$$

Substituting Eq. (13) into Eq. (14) yields

$$q=(k_f\ell\beta V\lambda/L)\cot(\beta V\lambda)F \tag{15}$$

If this value for q is substituted in Eq. (9), one obtains after simplifying

$$F''''+\alpha V^2F''-2\alpha\theta VF'+\beta V\lambda\alpha\cot(\beta V\lambda)F=0 \tag{16}$$

Thus, the problem has been reduced to the simultaneous solution of Eqs. (11) and (16) subject to conditions of continuity of W, W', and W'' at $\gamma=0$ and a discontinuity in W''' equal to the concentrated moving load. In addition, the condition of vanishing W at large distances from the load ($\gamma\to\pm\infty$) must be used.

Equation (11) has solutions of the form

$$F = e^{\pm i\lambda\gamma} \tag{17}$$

Substitution of Eq. (17) in Eq. (16) yields

$$\lambda^4 - \alpha V^2 \lambda^2 - 2\alpha\theta V\lambda i + \alpha\beta V\lambda \cot(\beta V\lambda) = 0 \tag{18}$$

where the positive exponent in Eq. (17) was used. Letting $\lambda = iy$, the characteristic equation can be rewritten in the form

$$y[y^3 + \alpha V^2 y + 2\alpha\theta V + \alpha\beta V \coth(\beta V y)] = 0 \tag{19}$$

Substitution of the boundary conditions

$$\begin{aligned} &\lim_{\epsilon\to 0}[W(0+\epsilon) - W(0-\epsilon)] = 0 \\ &\lim_{\epsilon\to 0}[W'(0+\epsilon) - W'(0-\epsilon)] = 0 \\ &\lim_{\epsilon\to 0}[W''(0+\epsilon) - W''(0-\epsilon)] = 0 \\ &\lim_{\epsilon\to 0}[W'''(0+\epsilon) - W'''(0-\epsilon)] = \alpha P \\ &\lim_{\gamma\to\pm\infty}[W(\gamma)] = 0 \end{aligned} \tag{20}$$

into the general solutions

$$W = \sum_{n=1}^{\infty} g_n F_n = \sum_{n=1}^{\infty} g_n e^{\lambda_n \gamma i}$$

$$U = \sum_{n=1}^{\infty} D_n g_n F_n = \sum_{n=1}^{\infty} g_n \frac{\sin[\beta V\lambda_n (1-\eta/L)]}{\sin(\beta V\lambda_n)} e^{\lambda_n \gamma i} \tag{21}$$

leads to the following expressions for the deflection of the beam W and the foundation displacement U on the right and the left of the load[18]

$$\frac{W_R}{r^2 p_0/EI} = e^{-a_1\gamma}\left[c_1 \sin(b_1\gamma) + c_2\cos(b_1\gamma)\right] \quad \gamma > 0 \tag{22}$$

$$\frac{W_L}{r^2 p_0/EI} = e^{a_2\gamma}\left[c_3 \sin(b_2\gamma) + c_2\cos(b_2\gamma)\right] \quad \gamma < 0 \tag{23}$$

$$\frac{U_R}{r^2 p_0/EI} = e^{-a_1\gamma} Re\left\{\frac{\sin[\beta V(1-\eta/L)(-b_1+ia_1)]}{\sin[\beta V(-b_1+ia_1]} \times (c_2+ic_1)[\cos(b_1\gamma) - i\sin(b_1\gamma)]\right\} \quad \gamma > 0 \tag{24}$$

$$\frac{U_L}{r^2 p_0/EI} = e^{a_2\gamma} Re\left\{\frac{\sin[\beta V(1-\eta/L)(-b_2-ia_2)]}{\sin[\beta V(-b_2-ia_2)]} \times (c_2+ic_3)[\cos(b_2\gamma) - i\sin(b_2\gamma)]\right\} \quad \gamma < 0 \tag{25}$$

In the above equations

$$c_1 = c_2[(a_1+a_2)^2 + b_2^2 - b_1^2]/[2b_1(a_1+a_2)]$$

$$c_2 = \frac{EI}{r^2 p_0} c_4$$

$$c_3 = c_2[-(a_1+a_2)^2 + b_2^2 - b_1^2]/[2b_2(a_1+a_2)]$$

$$c_4 = 2\alpha P(a_1+a_2)/\{[a_1(2a_1^2 + 3a_1a_2 + 2b_1^2) + a_2(3b_2^2 - b_1^2 - a_2^2)]2(a_1+a_2) + [3(a_1^2 - a_2^2) + (b_2^2 - b_1^2)][(b_2^2 - b_1^2) - (a_1+a_2)^2]\}$$

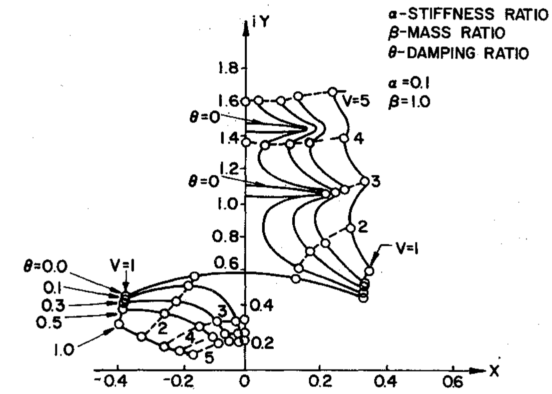

Fig. 2 Root loci for Eq. (18), $\beta = 1.0$.

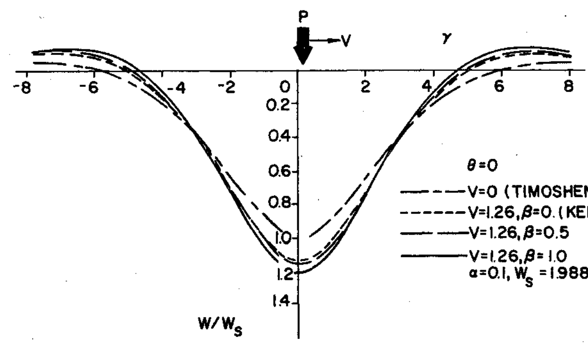

Fig. 3 Beam deflection—undamped system.

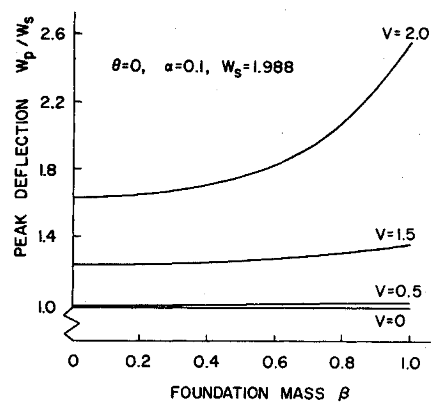

Fig. 4 Effect of foundation mass on beam deflection—undamped system.

$P = p_0/mr^2\Omega^2$ (dimensionless moving load) and $a_k + ib_k$ ($k = 1, 2$) is the kth root of Eq. (19).

IV. Numerical Results and Discussion

Computation of Roots of the Characteristic Equation

The transcendental nature of the characteristic equation (18) makes determination of the roots in closed form impossible. Two types of numerical methods were used to determine the roots approximately. The first method, based on the research of Muller[19] and Frank,[20] searches out roots by successive approximations of the characteristic equation with a second-degree polynomial with known roots. The second technique utilizes Newton's method to find new roots by taking small perturbations from a known root. In the undamped case, four complex roots and a countable set of real roots were found for velocities below critical. Above the critical velocity, all roots become real except for certain velocity ranges where complex roots appear again. For the damped case, both real and complex roots were determined for the complete range of velocities considered. Holder[18] discusses thoroughly the nature of the roots of the characteristic equation. Root locus plots for different foundation mass ratios and damping ratios are presented in Ref. 18. Figure 2 is an example of a root locus plot for $\alpha = 0.1$, $\beta = 1.0$. In this plot, the complex conjugates of the roots are not shown. Points representing the same velocity are joined by broken lines.

Undamped System Response

To determine the effect of foundation mass on the deflection shape, the beam displacement was computed for zero load velocity and for one-half the critical velocity for a massless foundation. Three different values of the ratio of foundation mass to beam mass were used in the latter case. Note that foundation mass is of no consequence in the static case. The results, normalized to the peak static deflection, are shown on Fig. 3. The static deflection is identical to that given by Timoshenko,[21] whereas the case of zero foundation mass agrees with that given by Kenney.[4] Note the symmetry of the deflection curves. This symmetry is not entirely obvious from Eqs. (22) and (23) unless it is observed that for the undamped case, $a_1 = a_2$ and $b_1 = b_2$. For the beam and foundation parameters noted on Fig. 3, it can be seen that the load velocity has a more pronounced effect on peak displacement than does foundation mass. The peak dynamic deflection is about 20% greater than the peak static deflection when the beam and foundation masses are the same ($\beta = 1$). When the foundation mass is neglected ($\beta = 0$), the dynamic amplification is reduced to about 14%. A more complete picture of the effect of foundation mass on beam peak displacement is given in Figs. 4 and 5. Figure 4 shows the ratio of peak dynamic deflection for a foundation-to-beam stiffness ratio of 0.1. For low load velocities, the usual assumption that the foundation mass has a negligible effect on peak displacement is seen to be valid. For example, at $V = 1.5$ increasing the mass ratio from 0 to 1 increases the dynamic-to-static displacement ratio about 10%. However, when the load velocity is increased to 2, the corresponding dynamic-to-static displacement ratio increases about 60%. This shows the importance of foundation mass if high load velocities are to be considered. Apparent in Fig. 4 is a strong dependence of dynamic displacement on load velocity. The results shown on Fig. 5 indicate a critical velocity, provided critical velocity is defined as the load velocity at which the deflection grows without bound. From Fig. 5, it can be seen that the effect of foundation mass is to reduce the value of the critical velocity. Attempts to compute beam and foundation response for load velocities above critical were not successful, although some complex roots of the characteristic equation appear for certain velocity ranges. It has been observed in the case of a massless foundation[4] that the response for velocities above critical cannot be determined without considering damping.

Damped System Response

When damping is considered, the deflection wave shape loses its symmetry and the peak dynamic deflection is

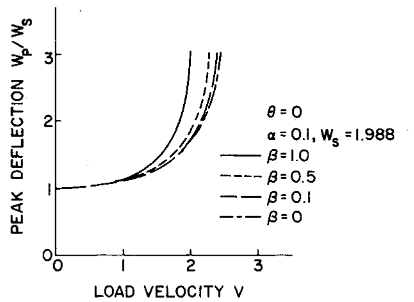

Fig. 5 Effect of local velocity on beam peak deflection—undamped system.

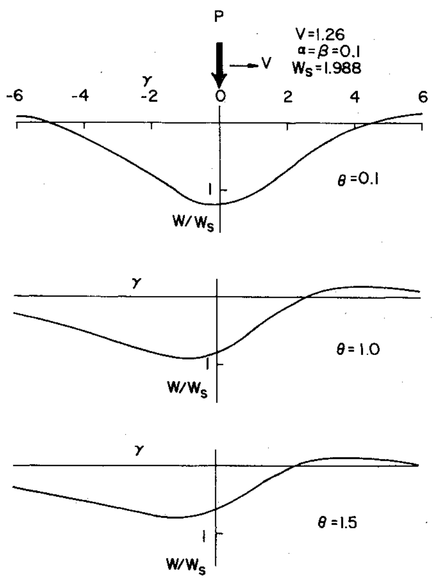

Fig. 6 Effect of damping on beam deflection.

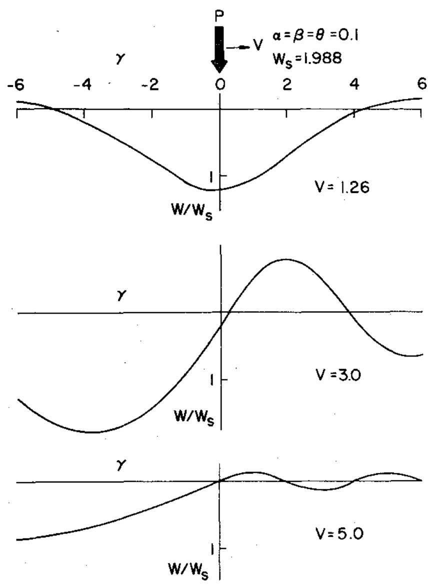

Fig. 7 Effect of load velocity on beam deflection.

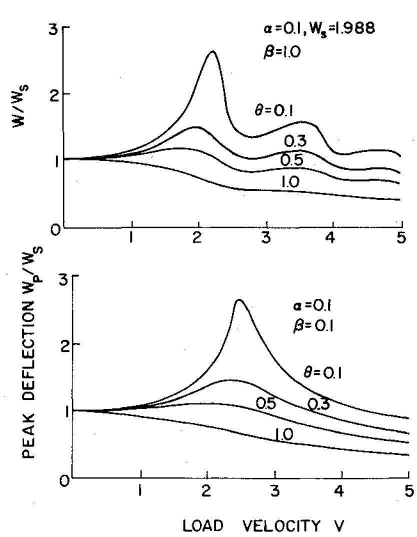

Fig. 8 Effect of load velocity on beam peak deflection.

reduced. These trends are depicted on Fig. 6. Also note that the peak deflection now occurs behind the moving load, which is always at $\gamma=0$. As damping is increased, the peak deflection moves further aft of the load. In addition, the frequency of the wave behind the load decreases. This trend continues until the frequency of the aft wave becomes zero; i.e., the wavelength becomes infinitely large. The damping corresponding to this change of character of the response can be defined as critical damping. The situation is similar to that for a massless foundation.[4] Examination of Eq. (23) shows that critical damping occurs when $b_2=0$. As might be expected, critical damping (and b_2) depends on α, β, and V and cannot be determined in closed form. However, root locus plots similar to those shown on Fig. 2 can be used to determine critical damping. One such plot was made for $\alpha=0.1$, $\beta=1.0$, $\theta=1.5$. When the velocity was increased to $V=1.4$, b_2 became zero.

The effect of load velocity on displacement was also investigated. Deflection wave shapes for load velocities below, near, and above critical are shown on Fig. 7. As the load velocity increases, the peak deflection moves aft of the load, the wavelength ahead of the load decreases, and the wavelength behind the load increases. As would be expected, the peak displacement occurs near the critical velocity of the undamped system. Figure 8 shows results of additional study of peak displacement for mass ratios of 0.10 and 1.0 and damping ratios ranging from 0.1 to 1.0. For $\beta=1.0$ (foundation mass per unit of beam length = beam mass per unit length), the foundation mass manifests itself by a series of peaks which occur at velocities greater than critical for the undamped system. These peaks might have been anticipated from the shape of the root locus plots on Fig. 2 for $\beta=1.0$. They are associated with the singularities introduced into the system by foundation mass and will be referred to hereafter as foundation resonance peaks. As the damping ratio is increased, all peaks including the one associated with critical velocity are reduced in magnitude. For $\beta=0.1$, no foundation resonance peaks are observed. This is because the lower mass system has resonance peaks at load velocities greater than those shown on Fig. 8.

The results shown on Fig. 9 give a better illustration of the effect of foundation mass on peak displacement. For small damping, the addition of foundation mass reduces the load velocity at which the maximum beam deflection occurs. In addition, foundation mass introduces small peaks in the deflection that occur at velocities higher than critical. Generally speaking, the magnitude of the peak deflection is not greatly affected by the addition of foundation mass. As damping is increased, the effect of foundation mass on peak displacement becomes very small.

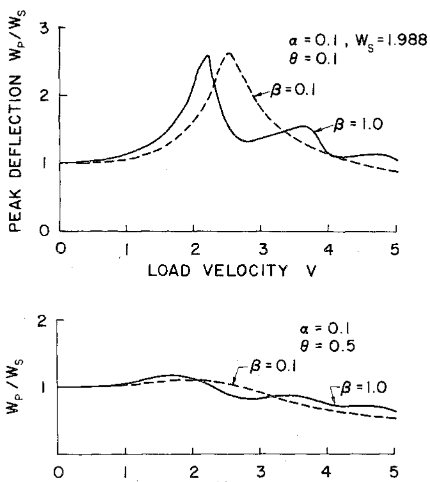

Fig. 9 Effect of foundation mass on beam peak deflection.

The foundation response was computed from Eqs. (24) and (25) for the undamped as well as the damped system. Numerical results are not presented here but are given in Ref. 18.

In conclusion, the results show that, for foundation-to-beam mass ratios approaching unity and load velocities near critical, foundation mass significantly affects the beam/foundation response. However, for small velocities or small foundation-to-beam mass ratios, foundation mass has negligible effect on the response.

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